DIMENSIONAL ANALYSIS

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\[
? \text{ ounces} = 31 \text{ pounds} \\
31 \text{ pounds} \times \frac{16 \text{ ounces}}{1 \text{ pound}} = 496 \text{ ounces}
\]
Objectives

- Understand the definition of units
- Add, subtract, multiply and divide numbers with identical units
- Convert one unit to another
- Multiply given information by conversion factors
Introduction

Most science courses require solving problems through mathematical operations. One of the most useful approaches to solving these problems is Dimensional Analysis.

This approach clearly:

1. **Identifies the “units” or “dimensions” on the numbers used in a mathematical operation**
2. **Indicates how the numbers in the problem should be combined (add, subtract, multiply, or divide)**
3. **Demonstrates whether the answer obtained is in the form requested by the problem**
Definition of Units

Unit is “a determinate quantity (as of length, time, heat, or value) adopted as a standard of measurement.” Unit defines what a number is while a number defines the magnitude of the unit.

Suppose you were asked “Do you have 5?” You would probably ask in return, “Five what?” It might be five cents, five dollars, five fingers, or even five pounds of salt. The number by itself is meaningless.

You have to say the quantity of whatever you are talking about and it is that whatever which is called the unit (or dimension) of the number.

In the above, it is the cents, dollars, fingers, or pounds which are the units.
Mathematical Manipulation of Units

In mathematical manipulations, units are treated in the same manner as numbers. For addition and subtraction, the old saying that you can't add apples to oranges applies. You can only add or subtract numbers with identical units.

For example:
1) 4 ft. + 5 ft. = 9 ft.

Both units are units of length so that the numbers can be added.
2) 4 ft. + 5 ft.² = ?

Since ft. is a unit of length and ft.², a unit of area, their addition is meaningless.

3) 4ft. + 5 in. = ?

Both are units of length, but since they are different, they cannot be added until one is converted to the same unit as the other one.
For multiplication and division units are treated in the same manner as numbers. Units are multiplied or divided just as numbers are multiplied or divided.

For example:
4) $4 \text{ cm} \times 2 \text{ g} = 8 \text{ g cm}$

5) $2 \text{ cm} \times 3 \text{ cm} = 6 \text{ cm}^2$
Mathematical Manipulation of Units

6) $4 \text{ cm} \times 10 \text{ mm} = ?$

Since both cm and mm are units of length, one should be converted to the other before multiplying.

This process will be shown in a later example.
Mathematical Manipulation of Units

7) \( \frac{4g}{2cm} = \frac{2g}{1cm} \) or \( 2g \text{ cm}^{-1} \)

A unit in the denominator can be placed in the numerator if the sign of the superscript is changed.

For example:

\[
\frac{1}{2cm} = \frac{1cm^{-1}}{2};
\]

\[
\frac{3g}{4cm^2} = \frac{3gcm^{-2}}{4}; \quad \frac{1g}{3m^{-2}} = \frac{1gm^2}{3}
\]
Mathematical Manipulation of Units

8) \( \frac{3 \, g \, cm}{1 \, cm} = 3 \, g \, cm \, cm^{-1} \) or \( 3 \, g \, cm \, cm^{-1} = 3 \, g \)

9) \( \frac{9 \, g \, cm^{-1}}{3 \, cm} = \frac{9}{3} \times \frac{g}{cm \, cm} = 3 \, g \, cm^{-1} \times \frac{1}{cm} = 3 \, g \, cm^{-1} \, cm^{-1} = 3 \, g \, cm^{-2} \)

or \( 3 \frac{g}{cm} = 3 \frac{g}{cm} \times \frac{1}{cm} = 3 \frac{g}{cm^2} \)
Mathematical Manipulation of Units

10) \[
\frac{4 \text{ sec.} \times 5 \text{ g cm}^{-1} \times 2 \text{ cm}}{4 \text{ cm}^2 \text{ sec.}^{-1}} = \frac{(4)(5)(2) \text{ sec. g cm}^{-1} \text{ cm}}{(4) \text{ cm}^2 \text{ sec.}^{-1}} =
\]

10 sec. g cm\(^{-1}\) cm \times \frac{1}{\text{cm}^2 \text{ sec.}^{-1}} = 10 \text{ sec. g cm}^{-1} \text{ cm cm}^{-2} \text{ sec.}

= 10 \text{ g sec.}^2 \text{ cm}^{-2}

or

\[
10 \text{ sec.} \frac{g}{cm} \text{ cm} \times 1/\frac{\text{cm}^2}{\text{sec.}} = 10 \text{ sec.} \frac{g}{cm} \text{ cm} \times \frac{\text{sec.}}{\text{cm}^2} = \frac{10 \text{ g sec.}^2}{\text{cm}^2}
\]
SECTION I PROBLEMS

Below are several practice problems. Work them out on a piece of scratch paper. The answers are at the end of the module.

11) 4 cm x 3 g cm =

12) 2 cm sec.\(^{-1}\) x 3 g sec.\(^{-2}\) =

13) \[
\frac{3 \text{ g cm sec.}^{-1} \times 4 \text{ cm sec.}^{-1}}{2 \text{ g cm}^{-3}} =
\]

14) \[
\frac{2 \text{ g} \times 3 \text{ cm}^2 \times 5 \text{ g cm}^{-3}}{3 \text{ sec.} \times 1 \text{ cm sec.}^{-1} \times 2 \text{ g cm}^{-2}} =
\]
Conversion Factors

When changing from one unit to another for the same quantity (length to length, cm to mm), a conversion factor is used.

Let's start with a very simple example of a conversion factor. We all know that there are 60 seconds in one minute. This really means that 60 seconds equal one minute. Mathematically this is:

60 seconds = 1 minute
Conversion Factors

If you divide both sides of an equality by the same thing, you still have equality. (This is a very important principle).

Take 60 seconds = 1 minute and divide both sides by 1 minute:

\[
\frac{60 \text{ seconds}}{1 \text{ minute}} = \frac{1 \text{ minute}}{1 \text{ minute}}
\]

\[
\frac{1 \text{ minute}}{1 \text{ minute}} = 1 \quad \text{(we say minutes “cancel”)}
\]

So, \[
\frac{60 \text{ seconds}}{1 \text{ minute}} = 1
\]
Conversion Factors

We call \( \frac{60\text{ seconds}}{1\text{ minute}} \) a conversion factor and it equals 1. All conversion factors equal 1.

You are probably more accustomed to saying “There are 60 seconds per minute.” The “per” means “divided by” and the number 1 is usually not written since it is understood.
Conversion Factors

Here are several other examples:

15) 24 hrs. = 1 day or 24 hrs./day or 24 hrs. day\(^{-1}\)

16) 100 cents = $1 or 100 cents/$ or 100 cents $^{-1}$

17) 2.54 cm = 1 in. or 2.54 cm/in. or 2.54 cm in.\(^{-1}\)
Conversion Factors

Let's go back to the first conversion factor:

\[60 \text{ sec.} = 1 \text{ minute}\]

Now divide both sides of the equality by 60 seconds:

\[
\frac{60 \text{ sec.}}{60 \text{ sec.}} = \frac{1 \text{ minute}}{60 \text{ sec.}}
\]

\[1 = \frac{1 \text{ minute}}{60 \text{ sec.}}\]
Conversion Factors

Since 1 min./60 sec. equals 1, it must also be a conversion factor. These two conversion factors, 60 sec./min. and 1 min./60 sec., are the same except one is the inverse of the other. Thus any conversion factor can be inverted and it will still be a conversion factor since the reciprocal of 1 is 1.

The above conversion factor can also be written as:

$$\frac{1 \text{ min.}}{60 \text{ sec.}} = \frac{1}{60} \text{ min./sec.} \text{ or } \frac{1}{60} \text{ min. sec.}^{-1}$$
Let's write the reciprocal of the previous examples on Slide 16:

15) 24 hrs./day or 1 day/24hrs. or 1/24 day hrs.$^{-1}$

16) 100 cents/$ or 1 dollar/100 cents or $1/100 cents^{-1}$

17) 2.54 cm./in. or 1 in./2.54 cm. or $\frac{1}{2.54} in. cm.^{-1}$
Conversion Factors

Many conversion factors have fixed definitions, such as, “There are 16 ounces per pound.”

\[ 16 \text{ oz.} = 1 \text{ lb.} \text{ or } 16\text{oz./lb.} \text{ or } 16 \text{ oz. lb.}^{-1} \]

Others are presented in a statement, such as, “A worker makes $7.50 per hour.”

\[ $7.50 = 1 \text{ hr.} \text{ or } $7.50/\text{hr.} \text{ or } $7.50 \text{ hr.}^{-1} \]

You should learn to spot these factors given in a problem.
Conversion Factors

Be careful when you write the dimensions of a number → a dimension must be sufficiently explicit so that you cannot confuse it with the dimension of any other number.

For example, if a problem says, “Onions cost 11 cents per pound,” you can write the conversion factor, 11 cents/lb. However, if in the same problem you have, “Potatoes cost 6 cents per pound,” then you must distinguish which vegetable you are talking about.

Thus, you should write the complete conversion factors: 11 cents/lb. onions and 6 cents/lb. potatoes.
Conversion factors do not necessarily have to be for a single hour, a single day, or a single pound.

For example, if a problem states that “An earth-mover can dig a trench 600 feet long in 3 hours,” the conversion would be:

\[
\frac{600 \text{ ft.}}{3 \text{ hrs.}} \text{ or } \frac{3 \text{ hrs.}}{600 \text{ ft.}}
\]

They may be used in this form or they can be written per hour or per foot by:

\[
\frac{600 \text{ ft.}}{3 \text{ hrs.}} = \frac{600}{3} \text{ ft. hr.}^{-1} = 200 \text{ ft. hr.}^{-1}
\]

\[
\frac{3 \text{ hrs.}}{600 \text{ ft.}} = \frac{3}{600} \text{ hr. ft.}^{-1} = 5 \times 10^{-3} \text{ hr. ft.}^{-1}
\]
SECTION II PROBLEMS

Here are more practice problems. For each problem write the conversion factor and its reciprocal in terms of per single unit. The answers are at the end of the module.

18) There are 12 inches in 1 foot.

19) My car used 9.0 gallons of gasoline to travel 270 miles.

20) He earned $24 in 8.0 hours of work.
Solving Problems

Now let's turn our attention to problem solving by dimensional analysis.

The following stepwise (4 steps) approach applies:

1. All problems ask a question, the answer of which will be a number and its dimensions.
2. All problems also have some information given in them which will also be a number and its dimensions.
Solving Problems

3. You will multiply the information given by conversion factors so that you cancel all dimensions except the dimensions you want in your answer.

4. You may multiply by as many conversion factors as you wish since all conversion factors equal 1, and multiplying something by 1 does not change its value.

(These steps are extremely important. Please be sure you have a thorough understanding of this problem solving approach.)
Solving Problems

Let's examine several examples:

21) How many seconds in 35 minutes?

The question asked is “How many seconds?” Therefore, when you get an answer it will have the dimension seconds. The information given in the problem is 35 minutes. You start by writing down the dimension of the answer so that you can check yourself at the end.

\[ ? \text{ sec.} = \]

Then you supply the information given:

\[ ? \text{ sec.} = 35 \text{ min.} \]
Solving Problems

Multiply the 35 min. by some conversion factor which will have the dimension minutes in the denominator (to cancel the unwanted minutes in the information given) and have the dimensions of the answer, seconds, in the numerator. Here we use the conversion factor 60 sec./min.

\[
? \text{ sec.} = 35 \text{ min.} \times 60 \frac{\text{sec.}}{\text{min.}} = 35 \times 60 \text{ sec.}
\]

\[= 2,100 \text{ sec.}\]

The unwanted dimension, minutes, cancels, and the only remaining dimension is seconds.
Solving Problems

You know that this must be the correct answer because the only dimension left is the dimension of the question asked, “How many seconds?”

Obviously, it would not work to multiply:

\[
35 \text{ min.} \times \frac{1 \text{ min.}}{60 \text{ sec.}}
\]

because none of the dimensions would cancel.
Solving Problems

22) How many hours are in six days?
The question asked is “How many hours?” The dimension of the answer, therefore, will be hours.

? hr. =
The information given is 6 days.

? hr. = 6 days
Solving Problems

Now, you want to multiply the information given by some conversion factor which will have the dimension **day** in the denominator (to cancel the unwanted dimension, **days**, in the information given). This should lead toward the dimension of the question asked, **hours**. Such a conversion factor is 24 hrs./day.

\[ \text{? \textit{hrs.}} = 6 \text{ \textit{days}} \times 24 \frac{\textit{hrs.}}{\textit{day}} = 144 \text{ \textit{hrs.}} \]

The unwanted dimension, **days**, cancels, leaving only the desired dimension, **hours**.
Multiplying by only one conversion factor usually will not get you all the way to the dimensions of the answer. However, since all conversion factors equal one, you can multiply by as many as you need. Now let's consider several examples requiring the use of multiple conversion factors.
23) How many seconds in three years?
The question asked is “How many seconds?” and the information given is 3 years.

\[ ? \text{ sec.} = 3 \text{ years} \]

Now you need some conversion factor which has years in the denominator (to cancel the years in the information given) and which will move you in the direction of the dimension of the question asked, seconds.

Such a conversion factor is: 365 days/yr.
Solving Problems

\[ ? \text{ sec.} = 3 \text{ yrs.} \times 365 \frac{\text{days}}{\text{yr.}} \]

The unwanted dimension, years, has been cancelled, but now you have the unwanted dimension \text{days} left. You need a conversion factor with days in the denominator to cancel that. The conversion 24 hrs./day will do just that:

\[ ? \text{ sec.} = 3 \text{ yrs.} \times 365 \frac{\text{days}}{\text{yr.}} \times 24 \frac{\text{hrs.}}{\text{day}} \]

The unwanted dimension is now \text{hours}. To move towards seconds, the next conversion factor would be 60 min./hr.
Solving Problems

\[ ? \text{ sec.} = 3 \text{ yrs.} \times \frac{365 \text{ days}}{\text{yr.}} \times 24 \frac{\text{hrs.}}{\text{day}} \times 60 \frac{\text{min.}}{\text{hr.}} \]

Finally, use the conversion factor 60 sec./min. to cancel the unwanted dimension minutes and leave seconds, the dimension of the question asked.

\[ ? \text{ sec.} = 3 \text{ yrs.} \times 365 \frac{\text{days}}{\text{yr.}} \times 24 \frac{\text{hrs.}}{\text{day}} \times 60 \frac{\text{min.}}{\text{hr.}} \times 60 \frac{\text{sec.}}{\text{min.}} \]

\[ ? \text{ sec.} = 3 \times 365 \times 24 \times 60 \times 60 \text{ sec.} = 9.5 \times 10^7 \text{ sec.} \]
24) How many miles in 2,640 yards?
The question is "How many miles?" and the information given is 2,640 yards.

\[ \text{? miles} = 2,640 \text{ yards} \]

Having the two conversion factors, 3 ft./yd. and 5,280 ft./mi. available, we proceed as follows:

\[ \text{? miles} = 2,640 \text{ yd.} \times 3 \frac{\text{ft.}}{\text{yd.}} \]

This gives the unwanted dimension feet. However, the conversion factor 5,280 ft./mi. does not have the dimension feet in the denominator for cancellation.
This problem can be solved in two ways. First, the above can be divided by the conversion factor 5,280 ft./mi.

\[
? \text{miles} = \frac{2,640 \text{ yd.} \times 3 \text{ ft./yd.}}{5,280 \text{ ft./mi.}}
\]

\[
= \frac{2,640 \text{ yd.} \times 3 \text{ ft.} \times \text{mi.}}{5,280 \text{ ft. yd.}}
\]

\[
= \frac{2,640 \times 3}{5,280} \text{ yd. ft. yd}^{-1} \text{ ft}^{-1} \text{ mi.}
\]

\[
? \text{miles} = 1.500 \text{ mi.}
\]
Solving Problems

Or second, $2,640 \text{ yd.} \times 3 \text{ ft./yd.}$ can be multiplied by the reciprocal of the conversion factor. The reciprocal of $5,280 \text{ ft./mi.}$ is $1 \text{ mi./5,280 ft.}$

$$? \text{ miles} = 2,640 \text{ yd.} \times \frac{3 \text{ ft.}}{\text{yd.}} \times \frac{1 \text{ mi.}}{5,280 \text{ ft.}}$$

$$= \frac{2,640 \times 3}{5,280} \text{ mi.}$$

$$? \text{ miles} = 1.500 \text{ mi.}$$
25) What is the cost of three shirts, if a box containing 12 shirts costs $27?

Here the conversion factors are contained in the problem and as a first step they should be written down.

\[
12 \text{ shirts/box and } $27/\text{box}
\]

Now, the question asked is “What is the cost?” which can be re-worded as, “How many dollars?”

\[
? \text{ dollars } =
\]
The information given is three shirts.

\[ \text{? dollars} = 3 \text{ shirts} \]

Applying the conversion factors, we have:

\[ \text{? dollars} = 3 \text{ shirts} \times \frac{\text{box}}{12 \text{ shirts}} \times \frac{\$27}{\text{box}} \]

\[ \text{? dollars} = 3 \times \frac{1}{12} \times \$27 = \$6.75 \text{ (for three shirts)} \]
Solving Problems

26) What is the gas consumption in miles per gallon of an automobile if it uses 0.1 gal. of gas in 100 sec. when traveling 60 miles/hr.

This problem is a little more difficult because the question asked, “How many miles per gallon?” contains two dimensions. Also, although two conversion factors can be written from the problem, 0.1 gal./100 sec.

AND

60 miles/hr.

the latter is treated as the information given.
Solving Problems

\[
? \text{miles/gal.} = \frac{60 \text{ miles}}{\text{hr.}} \times x
\]

Since the other conversion factor contains time in the dimension **seconds**, the dimension **hour** must be converted.

\[
? \text{miles/gal.} = \frac{60 \text{ miles}}{\text{hr.}} \times \frac{\text{hr.}}{60 \text{ min.}} \times \frac{\text{min.}}{60 \text{ sec.}} \times x
\]
Solving Problems

Now by multiplying by the reciprocal of the first conversion factor, the dimensions remaining after cancellation will be those of the question asked.

\[
? \text{miles/gal.} = \frac{60 \text{ miles}}{\text{hr.}} \times \frac{\text{hr.}}{60 \text{ min.}} \times \frac{\text{min.}}{60 \text{ sec.}} \times \frac{100 \text{ sec.}}{0.1 \text{ gal.}}
\]

\[
? \text{miles/gal.} = 60 \times \frac{1}{60} \times \frac{1}{60} \times \frac{100}{0.1} \text{ miles/gal.}
\]

\[
? \text{miles/gal.} = 16.7 \text{ miles/gal.}
\]
Here are several practice problems. The answers are at the end of the module.

27) How many seconds in 800 minutes?
28) How many dozens of eggs are there in 3,500 eggs?
29) How many minutes in five years?
30) What is the cost of six onions if three onions weigh 1.5 lb. and the price of onions is 11 cents per pound?
31) What is the cost to drive from Detroit to Washington, D.C. (524 miles) if the cost of gasoline is $1.02/gal. and the automobile gets 25 miles/gal.?

32) The price of a ream of paper is $4.00. There are 500 sheets of paper in a ream. If a sheet of paper weighs 0.500 oz., what is the price per pound of paper (16 oz./lb.)?

33) How many square inches in ten square centimeters? (2.54 cm./in.)
ANSWERS TO SECTION I PROBLEMS

11) \(4 \text{ cm} \times 3 \text{ g cm} = 12 \text{ g cm}^2\)

12) \(2 \text{ cm sec}^{-1} \times 3 \text{ g sec}^{-1} = 6 \text{ g cm sec.} \)

13) \(\frac{3g \text{ cm sec}^{-1} \times 4 \text{ cm sec}^{-1}}{2 \text{ g cm}^{-3}} = 6 \text{ g cm}^2 \text{ sec}^{-2} \text{ g}^{-1} \text{ cm}^3 = 6 \text{ cm}^5 \text{ sec}^{-2}\)

14) \(\frac{2 \text{ g} \times 3 \text{ cm}^2 \times 5 \text{ g cm}^{-3}}{3 \text{ sec} \times 1 \text{ cm sec}^{-1} \times 2 \text{ g cm}^{-2}} = 5 \text{ g cm}^2 \text{ g cm}^{-3} \text{ sec}^{-1} \text{ cm}^{-1} \text{ sec. g}^{-1} \text{ cm}^2 = 5g\)
ANSWERS TO SECTION II PROBLEMS

18) 12 in. = 1 ft.

\[
\frac{12 \text{ in.}}{1 \text{ ft.}} = \frac{12}{1} \text{ in./ft.} = 12 \text{ in./ft. OR } 12 \text{ in. ft.}^{-1}
\]

Reciprocal

\[
\frac{1 \text{ ft.}}{12 \text{ in.}} = \frac{1}{12} \text{ ft./in. OR } \frac{1}{12} \text{ ft. in.}^{-1}
\]
19) 270 miles = 9.0 gallons

\[
\frac{270 \text{ miles}}{9.0 \text{ gal.}} = \frac{270}{9} \text{ miles/gal.} = 30 \text{ miles/gal.}
\]

Reciprocal

\[
\frac{9.0 \text{ gal.}}{270 \text{ miles}} = \frac{9.0}{270} \text{ gal./mile} = 0.033 \text{ gal./mile}
\]
20) $24 = 8.0 \text{ hrs.}

\[
\frac{24}{8.0 \text{ hrs.}} = \frac{24}{8.0} \text{ hr.}^{-1} = 3.0 \text{ hr.}^{-1}
\]

Reciprocal

\[
\frac{8.0 \text{ hrs.}}{24} = \frac{8.0}{24} \text{ hrs.} \cdot \text{hr.}^{-1} = 0.33 \text{ hrs.} \cdot \text{hr.}^{-1}
\]
ANSWERS TO SECTION III
PROBLEMS

27) \( ? \text{ sec.} = 800 \text{ min.} \times 60 \frac{\text{sec.}}{\text{min.}} = 4.8 \times 10^4 \text{ sec.} \)

28) \( ? \text{ doz.} = 3,500 \text{ eggs} \times \frac{\text{doz.}}{12 \text{ eggs}} = 292 \text{ doz.} \)

29) \( ? \text{ min.} = 5 \text{ yrs.} \times 365 \frac{\text{days}}{\text{yr.}} \times 24 \frac{\text{hr.}}{\text{day}} \times 60 \frac{\text{min.}}{\text{hr.}} = 2.6 \times 10^6 \text{ min.} \)

27) \( ? \text{ cents} = 6 \text{ onions} \times 1.5 \frac{\text{lb.}}{\text{onions}} \times 11 \frac{\text{cents}}{\text{lb.}} = 33 \text{ cents} \)

28) \( ? \text{ dollars} = 524 \text{ miles} \times \frac{\text{gal.}}{25 \text{ miles}} \times \frac{$1.02}{\text{gal.}} = $21.38 \)
32) \[ \text{dollars/lb.} = \frac{\$4.00}{\text{ream}} \times \frac{\text{ream}}{500 \text{ shts.}} \times \frac{\text{sheet}}{0.500 \text{ oz.}} \times \frac{16 \text{ oz.}}{\text{lb.}} = 0.26 \text{ dol./lb.} \]

33) \( ? \text{ in.}^2 = 10 \text{ cm.}^2 \)

Deriving the conversion factor:

\[ 1 \text{ in.} = 2.54 \text{ cm.} \]
\[ (1\text{in.})^2 = (2.54 \text{ cm.})^2 \]

\[ \frac{1 \text{ in.}^2}{2.54^2 \text{ cm.}^2} = 1 \]

\[ ? \text{ in.}^2 = 10 \text{ cm.}^2 \times \frac{1 \text{ in.}^2}{2.54^2 \text{ cm.}^2} = 1.55\text{in.}^2 \]
If you have questions or are having difficulty solving these problems, ask the assistant in the Science Learning Center for help. If you feel confident in solving these problems ask the assistant for the posttest.